1995 MATH Challenge

1. You have your friend roll three dice, without showing you the outcome. Ask her to multiply the number on the first die by 5, then add 7 and double the result, add the number on the second die, multiply this result by 10, add the number on the third die, and tell you the result.

   a. If her result is 401, what were the numbers (in order) on the three dice?

   b. Explain how to determine in general the three numbers on the dice from the result on the prescribed calculation (and make clear why your method will work).

2. For real numbers $x > 1$ and integers $n > 1$, define

$$f_n(x) = \frac{n}{\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \ldots + \frac{1}{\log_n x}}$$

   Express

$$(n!)^{f_n(x)}$$

   as simply as possible in terms of x and n, and justify your answer. (But regard properties of the logarithm function as givens.)

3. a. If r and s are positive real numbers such that $\left(\sqrt{r} + \sqrt{s}\right)$ is rational, must $\sqrt{r}$ and $\sqrt{s}$ be rational? (Give a proof of your answer.)

   b. Suppose that each of r, s, and $\left(\sqrt{r} + \sqrt{s}\right)$ are rational. (r and s still assumed to be positive.) Prove that both $\sqrt{r}$ and $\sqrt{s}$ are rational.

4. Find, with proof, all real values of $a > 0$ such that

$$a^x \geq x^a \text{ holds for all } x > 0.$$

5. Through how many lattice points (*i.e.*, points (x,y) with x and y integers) does the curve

$$x^4 + y^4 - 16xy = 1$$

   pass?

6. When the number 1995^{1995} is expressed in decimal form, the sum of the digits is A, and when A is written in decimal form the sum of the digits is B. Let C be the sum of the digits when B is written in decimal form. Evaluate C. (Partial credit will be given for a reasonably good upper bound for C.)

7. At time t, particle 1 is in position $x_1 = 3\sin t$ and particle 2 is in position $x_2 = t(2 + \cos t)$ (both lie on the x-axis). Thus, at $t = 0$, both are at the origin. Are they ever again (*i.e.*, for some time $t > 0$) in the same position?

8. The sequence of functions $f_0, f_1, f_2, \ldots$ is defined recursively by

$$f_0 = 8, \text{ and for } n \geq 0, \quad f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}\ .$$

For each integer $n \geq 0$, find all real solutions of the equation $f_n(x) = 2x$.

9. Evaluate $\lim_{n \to \infty} \frac{1}{\sqrt[3]{n}} \sum_{k=1}^{n} \frac{1}{\sqrt[3]{(k+1)^2} + \sqrt[3]{k^2 - 1} + \sqrt[3]{(k-1)^2}}$.

(Hint: Note that the denominators have the form $x^2 + xy + y^2$. You should be able to take advantage of this to obtain a closed form for the sum.)

10. For $x > 0$, define the "continued exponential" function $f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ as follows:

$f_1(x) = x$, and for $n \geq 1$, $f_{n+1}(x) = x^{f_n(x)}$. Thus, $f_2(x) = x^x$, $f_3(x) = x^{x^x}$ (interpreted as $x^{\left(x^x\right)}$), etc. Now define

$$f(x) = \lim_{n \to \infty} f_n(x), \text{ when this limit exists.}$$

a. If $f(x_0) = 2$, find what x_0 must be, and explain why.

b. Note that part (a) does not assert that $f(x)$ ever has 2 as a value. For the number x_0 of part (a), prove that in fact $f(x_0) = 2$.