1997 MATH Challenge

1. How many "Friday the 13ths" are possible in a normal 365-day year? (Recall that April, June, September, and November each have 30 days, February has 28 days, and all other months have 31 days.)

2. Two people take turns cutting up a rectangular chocolate bar which is 6×8 squares in size. You are only allowed to cut the bar along a division between the squares, and your cut can only be a straight line. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. The last player who can (legally) break the chocolate wins (and gets to eat the chocolate bar). Is there a winning strategy for the first of second player. What about the general case (the starting bar is $m \times n$)?

3. Seventeen people are at a party. It turns out that for each pair of people present, exactly one of the following statements is always true: "They haven't met," "They are good friends," or "They hate each other." Prove that there must be a trio (3) of people, all of whom are either mutual strangers, mutual good friends, of mutual enemies.

4. Define f: $[0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ -2x+2 & \text{if } 1/2 < x \le 1. \end{cases}$$

Next, define a sequence f_n of functions from $[0,1]$ to $[0,1]$ as follows: Let $f_1 = f(x)$ and let $f_n(x) = f(f_{n-1}(x))$ for $n > 1$. Prove that for each n,

$$\int_0^1 f_n(x)dx = 1/2 .$$

5. A spherical, 3-dimensional planet has center (0,0,0) and radius 20. At any point (x,y,z) on the surface of this planet, the temperature is equal to $(x+y)^2 + (y-z)^2$ degrees. What is the average temperature of the surface of this planet?

6. (a) Color the plane in 2 colors. Prove that no matter how the coloration was done, there must be two points, exactly 1 mile apart, which are the same color.

   (b) In fact, show that the conclusion is still true, even if you use 3 colors!

   Note: The colorations are not necessarily continuous. For example, you may color all points (x,y) red if x is rational, y is rational; and all points blue is x is rational, y is irrational; and everything else is colored green.

7. Given n planets in space, where n is a positive integer. Each planet is a perfect sphere and all planets have the same radius R. Call a point on the surface of a planet *private* if it cannot be seen from any other planet. (Ignore things such as the height of people on the planet, clouds, perspective, etc. Also, assume that the planets are not touching each other.) It is easy to check that if $n = 2$, the total area is $4\pi R^2$, which is just the total area of one planet. Show that this is also true for all finite values of n.

8. Twenty-three people, each with integral weight, decide to play football, separating into two teams of eleven people, plus a referee. To keep things fair, the teams chosen must have equal t*otal* weight. It turns out that no matter who is chosen to be the referee, this can always be done. Prove that the 23 people must all have the same weight.

9. Let

$$\prod_{n=1}^{1997}\left(1+nx^{3^n}\right)=1+a_1x^{k_1}+a_2x^{k_2}+\ \ldots\ +a_mx^{k_m}$$

where $a_1,a_2,\ldots,a_m$ are nonzero and $k_1 < k_2 <\ \ldots\ < k_m$. Find a_{1997}.

10. There is one value of x between 0.6 and 1 satisfying the equation

$$5(\sqrt{1-x}+\sqrt{1+x})=6x+8\sqrt{1-x^2}\ .$$

Find it.